

## Change of principal rotation axis [by change of basis] – converting $k$ -hat to $j$ -hat

The basis vectors  $k$ -hat,  $j$ -hat, and  $i$ -hat are the bases of the **principal axes** for the **rectangular coordinate system**. The **principal singular complex numbers** for the rectangular coordinate system are the basis vectors appended with cross  $\times$  ( $k$ -cross,  $j$ -cross, and  $i$ -cross). From here on in [changed my mind on this], these will be written as  $k \times, j \times, i \times$  as the basis complex numbers... one can call them whatever they like [I had started out calling them curls since they discrete quarter curl around the basis vector]. I started calling them  $k$ -cross,  $j$ -cross, and  $i$ -cross because this is a very efficient way of doing the maths... unfortunate that it sounds racial... Here are some other possibilities as one may choose

$k$ -curl,  $k$ -mult(ibly),  $k$ -times,  $k$ -complex(number),  $xy$ -complex-number,  $k$ -imaginary number,  $k$ -cross-product,  $k$ -spin,  $k$ -rot(ation),  $k$ -hat-cross-product,  $k$ -imag(inary),  $k$ -hat-spin-generator,  $k$ -hat-spin-kernel,  $k$ -vector-product,  $k$ -hat-single-component-vector-product (imagine having to say that every time you had to write a complex number! even the simplest maths would take all day... lose my train of thought by the next line of working as well... the human mind only has so much forward looking RAM)...

... none of these quite say to me  $\hat{k} \times$  like  $k$ -cross.... its fast, its efficient, its meaningful (for cross-product around  $k$ -hat, a fixed principal axis of rotation) ... one thing I know for certain, this maths works. It is the necessary steps to extend Euler and Gauss, and meaningful as well... it brings the maths that they invented to the next level of advancement... mind you, once you really know what your doing, it will be possible to leave  $\times$  silent, as I will be discussing in a later document.... Dyads have been around for many years (many decades) but what I do is to take it to the next level with priming (which I called diads because triadic system is my goal), this is what has made this mathematical concept from relatively arcane obscurity to an overwhelmingly useful applied concept [soon to be as ubiquitous as complex numbers have been]. If nothing else, at minimum, this unifies the most sophisticated mathematics (that of complex analysis) with the most practical and applied (that of rotation matrices). This will increase the scope of complex analysis (to full  $R^3$  volumes) whilst increasing the sophistication of applied theory of linear algebra (most notably rotation matrices) with hundreds possibly thousands of additional equations to solve applied and theoretical scientific and engineering problems. As just one example, potentially converting all of Whittaker and Watson (Complex Analysis) over to the reals in a full  $R^3$  volume [three, rather than a single plane] and, further, extending this mathematics to matrix bases.

## Alternative mathematical name for $k$ -hat exponential, $j$ -hat exponential, and $i$ -hat exponential: **cisk**, **cisj**, and **cisi**

For a **standard rectangular basis** there are three complex exponentials, the  $k$ -hat exponential, the  $j$ -hat exponential, and the  $i$ -hat exponential, as well as the three corresponding conjugates [not including priming]. These terms are hard to pronounce whilst doing mathematics with them since they consist of six syllables: kay-hat-ex-pon-en-tial. The common alternative term for a complex exponential, which is a result of Euler's theorem, is cis. This is one syllable. This is a vernacular shortening of **c** for cos (for the **real part**), **i**-for **complex number**, and **s** for sine (for the **imaginary part**). This probably goes without saying for anyone reading this. cis-theta, for example, for an azimuthal argument of theta. From this document forward, cis as a basis-hat exponential synonymously means **circle**, that is, **circle in a real  $R^3$  vector volume**. Similarly as for sines and cosines for which there are three such forms (plus their scalars), there are also three forms of cis, those pertaining to each of the complex exponentials (basis-hat exponentials) on each of the three rectangular planes, each with the rotation axis relating to whether it is a  $k$ -hat,  $j$ -hat, or  $i$ -hat exponential. The new working nomenclature for these functions I am defining here [in good faith] in a similar way as for the sines and cosines. These are **cisk**, **ciskj**, and **cisi**, pronounced “sis-kay”, “sis-jay”, and “sis-eye”. For example, for an angular argument theta, these are pronounced “sis-kay-theta”, “sis-jay-theta”, and “sis-eye-theta”. These are just alternative ways of saying  $e^{\theta \hat{k} \times}$ ,  $e^{\theta \hat{j} \times}$ , and/or  $e^{\theta \hat{i} \times}$  concisely (four syllables, rather than eight) for the principal rectangular basis exponentials.

Once assigned a direction, the component sines and cosines (**sink** and **cosk** for **cisk** for example) revert to the scalar sines and cosines but the basis-hat cis function (**cisk** for example) retains the data on which rotation axis it pertains since the scalar components of the cis-theta is a circle on a specified plane perpendicular to the rotation axis (such as  $k$ -hat). Assigning the direction (for a non-degenerate circle) essentially means assigning a clockwise (**cw**) or counter-clockwise (**ccw**) direction of rotation. For example, as laboured upon earlier, assigning **cisk** to  $i$ -hat maps the circle to the  $x$ - $y$  plane in the counter-clockwise (**ccw**) direction whilst assigning it to  $j$ -hat maps the circle to the same plane but in the clockwise (**cw**) direction. Assigning it to  $k$ -hat is the degeneration of the cis function [the  $k$ -hat exponential] to the basis-hat itself since it is the rotation axis of the circle and the basis-hat is the eigenvector of the rotation axis. So again, once operated on the assigned directions, it becomes a circle with scalar sine and scalar cosine with a specified direction. So cis equivalently means circle (equivalently to the meaning of **c** for cosine, **i** for complex number, and **s** for sine): **cisk** means a circle on the  $x$ - $y$  plane (spinning around the  $z$ -axis, as a  $k$ -hat exponential [with  $k$ -cross complex number]), **ciskj** means a circle on the  $x$ - $z$  plane (spinning around the  $y$ -axis, as a  $j$ -hat exponential [with  $j$ -cross complex number]), and **cisi** means a circle on the  $y$ - $z$  plane (spinning around the  $x$ -axis, as a  $i$ -hat exponential [with  $i$ -cross complex number]).

I will show in the next several pages [in another document] that coordinate systems other than the rectangular system of  $i$ -hat,  $j$ -hat, and  $k$ -hat, notably cylindrical and spherical, are **cis-scalings** ( $k$ -hat and  $j$ -hat exponential scalings) of the **standard rectangular system** of  $i$ -hat,  $j$ -hat, and  $k$ -hat.

Another notation I have started using is **expi**, **expj**, and **expk**. This is equivalent to **cisi**, **cisj**, and **cisk**. Throughout this document I preferentially use **expi**, **expj**, and **expk** when referring to basis exponentials and **cisi**, **cisj**, and **cisk** when stressing the component basis harmonics of **cosi**, **sini**, **cosj**, **sinj**, **cosk**, and **sink**. I also use the equalities of **expi**  $\theta$  = **cisi**  $\theta$ , **expj**  $\theta$  = **cisj**  $\theta$ , and

**expk**  $\theta$  = **cisk**  $\theta$  to succinctly emphasize Euler's formulae linking complex [basis] exponentials to [basis] harmonics, and powers of these for de-Moivre's theorem with **(expi  $\theta$ )**<sup>n</sup> = **cisi**  $n\theta$ , **(expj  $\theta$ )**<sup>n</sup> = **cisj**  $n\theta$ , and **(expk  $\theta$ )**<sup>n</sup> = **cisk**  $n\theta$ .

## Introduction

There are six main basis sines and cosines that have been introduced and defined, **sink**, **sinj**, and **sini**, and **cosk**, **cosj**, and **cosi**. If one takes a spool, for example, around which wire is round and the rotation axis is  $k$ -hat, it is possible to physically rotate the entire spool around the  $i$ -hat axis. So then the question is, is the spool still a  $k$ -hat sine,  $k$ -hat cosine, and  $k$ -hat exponential (**sink**, **cosk**, and **expk**)? The answer to this is no and yes. It is no longer a  $k$ -cross basis sine, basis cosine, and basis exponential, but rather a  $j$ -cross basis sine, basis cosine, and basis exponential. Yes, it is still a basis function but defined for a different principal axis (so **different but the same**... the linear algebra term for this is **similarity**). This is a change of basis. This change of basis converts the  $k$ -hat basis sine (**sink**),  $k$ -hat basis cosine (**cosk**), and  $k$ -hat basis exponential (**expk**) to that axis. For example, a 90 degree rotation around the  $i$ -hat axis is a  $\pi/2$   $i$ -hat exponential basis change of  $k$ -hat to that of  $j$ -hat. The change of matrix basis converts the  $k$ -hat basis sine (**sink**),  $k$ -hat basis cosine (**cosk**), and  $k$ -hat exponential (**expk**) to that of the  $j$ -hat principal axis for  $j$ -hat basis sine (**sinj**),  $j$ -hat basis cosine (**cosj**), and  $j$ -hat basis exponential (**expj**). This is the same rotation as for a Rubiks cube, but applied as a change of basis [to both vector bases of the outer product] to the matrix basis-functions rather than, say, an operator changing the coordinates of a position vector or the orientation of a colour vector. For the  $k$ -hat exponential and basis harmonics (**expk**=**cisk**), the change of basis is to the  $j$ -hat basis exponential (**expj**) is **expj**( $\pi/2$ ). This works by change of [rotation axis] basis of the component vector bases in the matrix basis. The change of basis is applied as a conjugate from the left (unprimed left to right) and as a non-conjugate from the right (primed from right to left) on the basis function of the principal axis that is being changed, here the  $k$ -hat axis. The change of basis is applied in such a way because a matrix operator is the diadic expansion as outer-product of two vector bases, each of which requires the base change, one of which being the other's conjugate – this is why there is a conjugate from the left and non-conjugate from the right [or visa-versa] as will be shown, argued, and proved mathematically throughout the remainder of this and subsequent documents.

## Change of principal axis by change of basis

Here I prove change of basis from principal  $k$ -hat basis functions (in  $k \times$ ) to principal  $j$ -hat basis functions (in  $j \times$ ) with the  $i$ -hat  $\pi/2$  exponential **expi**( $\pi/2$ ) for the following functions

### 1. $k$ -hat basis complex number ( $k$ -cross)

$$\hat{k} \times \rightarrow \hat{j} \times$$

### 2. powers of $k$ -hat basis complex number

$$(\hat{k} \times)^n \rightarrow (\hat{j} \times)^n$$

### 3. general arbitrary polynomial of $k$ -hat basis complex number

$$p(\hat{k} \times) \rightarrow p(\hat{j} \times)$$

### 4. $k$ -hat exponential as polynomial expansion of $k$ -cross

$$\text{expk} \theta \rightarrow \text{expj} \theta \quad (e^{\theta \hat{k} \times} \rightarrow e^{\theta \hat{j} \times})$$

### 5. $k$ -hat basis cosine and $k$ -hat basis sine

$$\text{cosk} \theta \rightarrow \text{cosj} \theta \quad \text{and} \quad \text{sink} \theta \rightarrow \text{sinj} \theta$$

### 6. powers of $k$ -hat basis exponential (de-Moivre's theorem)

$$(\text{expk} \theta)^n = \text{cisk} n\theta \rightarrow (\text{expj} \theta)^n = \text{cisj} n\theta$$

Note that for all of the following mathematics, what has been proved is that the change of basis has changed the principal  $k$ -hat axis to principal  $j$ -hat axis for the various basis functions. What is also proved in doing so is that each of the various basis functions dependent on one principal axis is formally similar (in the linear algebra sense) to the basis function on either of the other principal axes in the rectangular (xyz) system. Hence it is two proofs in one for each of the basis functions covered [change of basis and similarity]. Each of these eight relations are proved

$$(\hat{i} \hat{i}' - \hat{i} \times)(\hat{k} \times)(\hat{i} \hat{i}' + \hat{i} \times) = \hat{j} \times$$

$$e^{-(\pi/2)\hat{i} \times}(\hat{k} \times)e^{(\pi/2)\hat{i} \times} = \hat{j} \times$$

$$(\hat{i} \hat{i}' - \hat{i} \times)(\hat{k} \times)^n(\hat{i} \hat{i}' + \hat{i} \times) = (\hat{j} \times)^n$$

$$e^{-(\pi/2)\hat{i} \times}(\hat{k} \times)^n e^{(\pi/2)\hat{i} \times} = (\hat{j} \times)^n$$

$$(\hat{i} \hat{i}' - \hat{i} \times)p(\hat{k} \times)(\hat{i} \hat{i}' + \hat{i} \times) = p(\hat{j} \times)$$

$$e^{-(\pi/2)\hat{i} \times}p(\hat{k} \times)e^{(\pi/2)\hat{i} \times} = p(\hat{j} \times)$$

$$(\hat{i} \hat{i}' - \hat{i} \times)(e^{\theta \hat{k} \times})(\hat{i} \hat{i}' + \hat{i} \times) = e^{\theta \hat{j} \times}$$

$$e^{-(\pi/2)\hat{i} \times}(e^{\theta \hat{k} \times})e^{(\pi/2)\hat{i} \times} = e^{\theta \hat{j} \times}$$

$$(\hat{i} \hat{i}' - \hat{i} \times)(\text{cosk} \theta + (\text{sink} \theta) \hat{k} \times)(\hat{i} \hat{i}' + \hat{i} \times) = \text{cosj} \theta + (\text{sinj} \theta) \hat{j} \times$$

$$e^{-(\pi/2)\hat{i} \times}(\text{cosk} \theta)e^{(\pi/2)\hat{i} \times} = \text{cosj} \theta$$

$$(\hat{i} \hat{i}' - \hat{i} \times)(\text{cosk} \theta)(\hat{i} \hat{i}' + \hat{i} \times) = \text{cosj} \theta$$

$$e^{-(\pi/2)\hat{i} \times}(\text{sink} \theta) \hat{k} \times e^{(\pi/2)\hat{i} \times} = (\text{sinj} \theta) \hat{j} \times$$

$$(\hat{i} \hat{i}' - \hat{i} \times)(\text{sink} \theta) \hat{k} \times(\hat{i} \hat{i}' + \hat{i} \times) = (\text{sinj} \theta) \hat{j} \times$$

$$(\hat{i} \hat{i}' - \hat{i} \times)(\text{sink} \theta)(\hat{i} \hat{i}' + \hat{i} \times) = \text{sinj} \theta$$

$$e^{-(\pi/2)\hat{i} \times}(\text{sink} \theta)e^{(\pi/2)\hat{i} \times} = \text{sinj} \theta$$

which hence also proves that basis functions on different principal axes are similar. Note that the ‘*similarity*’ implied here is not for diagonal matrices or Jordan forms so is a slight expansion of what it means to state that two functions [that can be diadically expanded into matrix bases] are similar.

### 1. $\hat{k} \times$ basis complex number $\hat{k} \times \rightarrow \hat{j} \times$

This proof is the most important, since once the change of basis is proved for the single component cross-product ( $k$ -cross,  $\hat{k} \times$ , the  $k$ -hat complex number) it is easily proved for all of the basis functions of  $\hat{k} \times$  as power series expansions of  $\hat{k} \times$ . For  $k$ -hat complex number,  $\hat{k} \times$ , changing the principal axis from  $k$ -hat to  $j$ -hat with the  $\pi/2$   $i$ -hat change of basis

$$\exp\left(-\frac{1}{2}\pi \hat{i} \times\right) \left(\hat{k} \times\right) \exp\left(\frac{1}{2}\pi \hat{i} \times\right) (\cdot) = (\hat{i} \hat{i}' - \hat{i} \times) \left(\hat{k} \times\right) (\hat{i} \hat{i}' + \hat{i} \times) (\cdot) \quad \text{where } \mathbf{cisi}(\pm\pi/2) = \exp\left(\pm\frac{1}{2}\pi \hat{i} \times\right) = \hat{i} \hat{i}' + \hat{i} \times$$

In the  $i$ -hat direction  $(\hat{i} \hat{i}' - \hat{i} \times) \left(\hat{k} \times\right) (\hat{i} \hat{i}' + \hat{i} \times) \hat{i}$

$$\begin{aligned} (\hat{i} \hat{i}' - \hat{i} \times) \left(\hat{k} \times\right) (\hat{i} \hat{i}' + \hat{i} \times) \hat{i} &= (\hat{i} \hat{i}' - \hat{i} \times) \left(\hat{k} \times\right) (\hat{i} \hat{i}' \hat{i} + \hat{i} \times \hat{i}) \\ &= (\hat{i} \hat{i}' - \hat{i} \times) \left(\hat{k} \times\right) (\hat{i} 1 + \hat{i} 0) \\ &= (\hat{i} \hat{i}' - \hat{i} \times) \hat{k} \times \hat{i} \\ &= (\hat{i} \hat{i}' - \hat{i} \times) (\hat{j}) \\ &= \hat{i} \hat{i}' \hat{j} + \hat{i} \times \hat{j} \\ &= \hat{i} 0 + \hat{k} \\ &= \hat{k} \\ &= (\hat{j} \times) \hat{i} \end{aligned}$$

In the  $j$ -hat direction  $(\hat{i} \hat{i}' - \hat{i} \times) \left(\hat{k} \times\right) (\hat{i} \hat{i}' + \hat{i} \times) \hat{j}$

$$\begin{aligned} (\hat{i} \hat{i}' - \hat{i} \times) \left(\hat{k} \times\right) (\hat{i} \hat{i}' + \hat{i} \times) \hat{j} &= (\hat{i} \hat{i}' - \hat{i} \times) \left(\hat{k} \times\right) (\hat{i} \hat{i}' \hat{j} + \hat{i} \times \hat{j}) \\ &= (\hat{i} \hat{i}' - \hat{i} \times) \left(\hat{k} \times\right) (\hat{i} 0 + \hat{k}) \\ &= (\hat{i} \hat{i}' - \hat{i} \times) \hat{k} \times \hat{k} \\ &= (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} 0) \\ &= 0 \\ &= (\hat{j} \times) \hat{j} \end{aligned}$$

In the  $k$ -hat direction  $(\hat{i} \hat{i}' - \hat{i} \times) \left(\hat{k} \times\right) (\hat{i} \hat{i}' + \hat{i} \times) \hat{k}$

$$\begin{aligned} (\hat{i} \hat{i}' - \hat{i} \times) \left(\hat{k} \times\right) (\hat{i} \hat{i}' + \hat{i} \times) \hat{k} &= (\hat{i} \hat{i}' - \hat{i} \times) \left(\hat{k} \times\right) (\hat{i} \hat{i}' \hat{k} + \hat{i} \times \hat{k}) \\ &= (\hat{i} \hat{i}' - \hat{i} \times) \left(\hat{k} \times\right) (\hat{i} 0 + \hat{j}) \\ &= (\hat{i} \hat{i}' - \hat{i} \times) \hat{k} \times \hat{j} \\ &= (\hat{i} \hat{i}' - \hat{i} \times) (-\hat{i}) \\ &= \hat{i} \hat{i}' \hat{i} + \hat{i} \times \hat{i} \\ &= \hat{i} 1 + \hat{i} 0 \\ &= \hat{i} \\ &= (\hat{j} \times) \hat{k} \end{aligned}$$

Hence, the  $\pi/2$   $i$ -hat change of basis of principal  $k$ -hat axis complex number  $k$ -cross ( $\hat{k} \times$ ) equals the principal  $j$ -hat axis complex number  $j$ -cross ( $\hat{j} \times$ ), that is,

$$(\hat{i} \hat{i} - \hat{i} \times)(\hat{k} \times)(\hat{i} \hat{i} + \hat{i} \times)(\cdot) = (\hat{j} \times)(\cdot), \text{ for any basis direction } (\cdot) = \hat{i}, \hat{j}, \hat{k}.$$

Hence, this method of **change of principal axis**, and the notions of **similarity** show that it is possible to convert the complex numbers [basis-crosses, that is, **single component basis-cross-products**] from one principal axis to another similarly as it is possible to convert the harmonic functions (as functions of a principal basis-cross) between principal axes. The method is the same as for the basis-exponentials as discussed shortly. This proof is easily demonstrated, as an alternative to basis cross-products (complex numbers), with the diadically expanded forms of basis exponentials (rotation matrices).

By this method, to convert  $k$ -cross ( $k \times$ ) to  $j$ -cross ( $j \times$ ), ones makes the change of basis with the  $\pi/2$   $i$ -hat exponential expressed in diadically matrix expanded form

$$\mathbf{cisi}(-\pi/2) (\hat{k} \times) \mathbf{cisi}(\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \hat{j} \times \quad \text{where } \mathbf{cisi}(\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

therefore  $\mathbf{cisi}(-\pi/2) (\hat{k} \times) \mathbf{cisi}(\pi/2) = \hat{j} \times$  and, further,  $k$ -cross is proved similar to  $j$ -cross  $\hat{k} \times \sim \hat{j} \times$ . Note **cisi=expi** is some new notation I have started to include in my work.

## 2. powers of $k$ -hat basis complex number

$$(\hat{k} \times)^n \rightarrow (\hat{j} \times)^n$$

This proof is the same as that for powers of a diagonalized matrix in an eigenvalue problem, and is essentially what this maths does but in terms of a mathematical argument based on what was hence previously thought of as complex numbers ( $i = k \times$ ): Most easily seen in reverse, which I'll do first, and then reverse it back as the proof

$$\begin{aligned} (\hat{j} \times)^n &= \left[ (\hat{i} \hat{i} - \hat{i} \times)(\hat{k} \times)(\hat{i} \hat{i} + \hat{i} \times) \right]^n \\ &= \prod_1^n (\hat{i} \hat{i} - \hat{i} \times)(\hat{k} \times)(\hat{i} \hat{i} + \hat{i} \times) \\ &= \left[ (\hat{i} \hat{i} - \hat{i} \times)(\hat{k} \times)(\hat{i} \hat{i} + \hat{i} \times) \right] \left[ (\hat{i} \hat{i} - \hat{i} \times)(\hat{k} \times)(\hat{i} \hat{i} + \hat{i} \times) \right] \left[ (\hat{i} \hat{i} - \hat{i} \times)(\hat{k} \times)(\hat{i} \hat{i} + \hat{i} \times) \right] \left[ (\hat{i} \hat{i} - \hat{i} \times)(\hat{k} \times)(\hat{i} \hat{i} + \hat{i} \times) \right] \\ &= (\hat{i} \hat{i} - \hat{i} \times)(\hat{k} \times) \cancel{(\hat{i} \hat{i} + \hat{i} \times)(\hat{i} \hat{i} - \hat{i} \times)} (\hat{k} \times) \cancel{(\hat{i} \hat{i} + \hat{i} \times)(\hat{i} \hat{i} - \hat{i} \times)} (\hat{k} \times) \dots (\hat{k} \times) \cancel{(\hat{i} \hat{i} + \hat{i} \times)(\hat{i} \hat{i} - \hat{i} \times)} (\hat{k} \times) (\hat{i} \hat{i} + \hat{i} \times) \\ &= (\hat{i} \hat{i} - \hat{i} \times)(\hat{k} \times)(\hat{k} \times)(\hat{k} \times) \dots (\hat{k} \times)(\hat{k} \times)(\hat{i} \hat{i} + \hat{i} \times) \quad \text{since } (\hat{i} \hat{i} + \hat{i} \times)(\hat{i} \hat{i} - \hat{i} \times) = I \\ &= (\hat{i} \hat{i} - \hat{i} \times) \left( \prod_1^n (\hat{k} \times) \right) (\hat{i} \hat{i} + \hat{i} \times) \\ &= (\hat{i} \hat{i} - \hat{i} \times) (\hat{k} \times)^n (\hat{i} \hat{i} + \hat{i} \times) \end{aligned}$$

since  $(\hat{i} \hat{i} - \hat{i} \times)(\hat{i} \hat{i} + \hat{i} \times) = \exp(-\frac{1}{2} \pi \hat{i} \times) \exp(\frac{1}{2} \pi \hat{i} \times) = I$ . One can read the power of  $j$ -cross ( $j \times$ ) to the change of basis power of  $k$ -cross ( $k \times$ ) in reverse, which then proves that  $(\hat{i} \hat{i} - \hat{i} \times)(\hat{k} \times)^n (\hat{i} \hat{i} + \hat{i} \times) = (\hat{j} \times)^n$  so that the change of principal  $k$ -hat axis complex number  $k$ -cross ( $k \times$ ) to the principal  $j$ -hat axis complex number  $j$ -cross ( $j \times$ ) holds equally for arbitrary powers.

$$\begin{aligned}
& (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times)^n (\hat{i} \hat{i}' + \hat{i} \times) \\
&= (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times) (\hat{k} \times) (\hat{k} \times) \dots (\hat{k} \times) (\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) \\
&= (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times) I (\hat{k} \times) I (\hat{k} \times) I \dots I (\hat{k} \times) I (\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) \quad \text{and with } (\hat{i} \hat{i}' + \hat{i} \times) (\hat{i} \hat{i}' - \hat{i} \times) = I \\
&= (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times) (\hat{i} \hat{i}' - \hat{i} \times) (\hat{i} \hat{i}' + \hat{i} \times) (\hat{k} \times) (\hat{i} \hat{i}' - \hat{i} \times) (\hat{i} \hat{i}' + \hat{i} \times) (\hat{k} \times) \dots (\hat{k} \times) (\hat{i} \hat{i}' - \hat{i} \times) (\hat{i} \hat{i}' + \hat{i} \times) (\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) \\
&= \left[ (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) \right] \left[ (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) \right] \left[ (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times) \dots (\hat{k} \times) (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) \right] \\
&= \left[ (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) \right]^n \quad \text{and since } (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) = \hat{j} \times \\
&= (\hat{j} \times)^n
\end{aligned}$$

Note that while this is the same mathematics of why diagonalized matrices can be raised to powers,  $k$ -cross ( $k \times$ ) is not diagonal nor is it in Jordan form and nor is  $j$ -cross ( $j \times$ ). This means that this is additional degrees of freedom for a mathematician's tool kit and an expansion of mathematical understanding.

So raising to powers also holds for the complex numbers themselves which means that the quartic roots of unity on any plane hold via similarity to any of the other planes in  $R^3$ . For the example, since  $\text{cisi}(-\pi/2) (\hat{k} \times) \text{cisi}(\pi/2) = \hat{j} \times$ , which is easily proved alternatively by the complex numbers and basis exponentials in matrix bases. Since  $k$ -cross is the singular complex number so that once diadically expanded into a matrix basis [the quarter turn rotation matrix without the eigenvalue] is a **periodic matrix** with period 4, similar as its function as the complex number  $(i, i^2, i^3, i^4 = k \times, k \times^2, k \times^3, k \times^4)$ . This is similarly with that to which has had its base changed to  $j$ -cross  $(i, i^2, i^3, i^4 = j \times, j \times^2, j \times^3, j \times^4)$  raised to the same power. So it suffices to show that raised up to and including power 4 returns the equivalent power in the  $j$ -hat complex number  $j$ -cross. This is the identity without the rotational eigenvalue [without the  $kk'$  for  $k \times^4$  and without the  $jj'$  for  $j \times^4$ ]. Note that  $(i, i^2, i^3, i^4 = k \times, k \times^2, k \times^3, k \times^4)$  to  $(i, i^2, i^3, i^4 = j \times, j \times^2, j \times^3, j \times^4)$  is the base-change.

First order was the last proof. Second power

$$e^{-(\pi/2)\hat{i} \times} (\hat{k} \times)^2 e^{(\pi/2)\hat{i} \times} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (\hat{j} \times)^2$$

third power

$$e^{-(\pi/2)\hat{i} \times} (\hat{k} \times)^3 e^{(\pi/2)\hat{i} \times} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (\hat{j} \times)^3$$

four power

$$e^{-(\pi/2)\hat{i} \times} (\hat{k} \times)^4 e^{(\pi/2)\hat{i} \times} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\hat{j} \times)^4$$

hence  $e^{-(\pi/2)\hat{i} \times} (\hat{k} \times)^n e^{(\pi/2)\hat{i} \times} = (\hat{j} \times)^n$  for all integer  $n$  [it holds for any real  $n$  although not proved here]. Note that this is equivalent to

$2\pi \cdot n$  for a Fourier transform, basis exponentials repeat after  $2\pi$  periods whilst in the matrix basis form this period is integers similarly as  $i = k \times$  in complex algebra, that is, integer multiples of 4 [group order of  $k$ -cross] times  $\pi/2$  [angular argument of the  $i$ -hat change of basis matrix].

### 3. general arbitrary polynomial of $k$ -hat basis complex number

$$\mathbf{p}(\hat{k} \times) \rightarrow \mathbf{p}(\hat{j} \times)$$

Once this result is proved, that for an arbitrary polynomial  $\mathbf{p}$  in  $\hat{k} \times$ , the results for a change of basis of a basis function of one principal axis to another then holds for any of the basis functions, most importantly, basis sines and cosines (**sini** and **cosi**, **sinj** and **cosj**, **sink** and **cosk**) and basis exponentials (**expi**, **expj**, **expk**) and their powers and hence for a change of basis of de-Moivre's theorem from one principal axis to another ( $(\mathbf{expk}\theta)^n = \mathbf{cisk} n\theta \rightarrow (\mathbf{expj}\theta)^n = \mathbf{cisj} n\theta$ ). It also holds for the other basis functions of tangent (**tani**, **tanj**, and **tank**) and cotangent (**coti**, **cotj**, and **cotk**), secant (**seci**, **secj**, and **seck**), and so on, as well as the **polar**

**cylindrical complex numbers** and **polar spherical complex numbers** and harmonics (which I will get ready after this document... trying to make sure it reads well takes the most time).

For an arbitrary polynomial in  $k$ -cross ( $k \times$ ), that is, a polynomial of arbitrary coefficients and arbitrary polynomial order

$$\mathbf{p}(\hat{k} \times) = a_0 + a_1(\hat{k} \times) + a_2(\hat{k} \times)^2 + a_3(\hat{k} \times)^3 + a_4(\hat{k} \times)^4 + \dots$$

Applying the change of basis for a change of principal  $k$ -hat axis (for  $k \times$ ) to the principal  $j$ -hat axis (for  $j \times$ ), that is from  $\mathbf{p}(k \times)$  to  $\mathbf{p}(j \times)$ ,

$$\mathbf{p}(\hat{j} \times) = a_0 + a_1(\hat{j} \times) + a_2(\hat{j} \times)^2 + a_3(\hat{j} \times)^3 + a_4(\hat{j} \times)^4 + \dots$$

with the  $\pi/2$   $i$ -hat exponential  $\mathbf{expi}(\pi/2) = \mathbf{cisi}(\pi/2)$  and its conjugate  $\mathbf{expi}(-\pi/2) = \mathbf{cisi}(-\pi/2)$

$$\exp\left(-\frac{1}{2}\pi \hat{i} \times\right) \mathbf{p}(\hat{k} \times) \exp\left(\frac{1}{2}\pi \hat{i} \times\right) (\cdot) = (\hat{i} \hat{i}' - \hat{i} \times) \mathbf{p}(\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) (\cdot) \quad \text{where } \mathbf{cisi}(\pm\pi/2) = \exp\left(\pm\frac{1}{2}\pi \hat{i} \times\right) = \hat{i}' \hat{i} + \hat{i} \times$$

$$\begin{aligned} (\hat{i} \hat{i}' - \hat{i} \times) \mathbf{p}(\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) &= (\hat{i} \hat{i}' - \hat{i} \times) (a_0 + a_1(\hat{k} \times) + a_2(\hat{k} \times)^2 + a_3(\hat{k} \times)^3 + a_4(\hat{k} \times)^4 + \dots) (\hat{i} \hat{i}' + \hat{i} \times) \\ &= (\hat{i} \hat{i}' - \hat{i} \times) a_0 (\hat{i} \hat{i}' + \hat{i} \times) + (\hat{i} \hat{i}' - \hat{i} \times) a_1(\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) + (\hat{i} \hat{i}' - \hat{i} \times) a_2(\hat{k} \times)^2 (\hat{i} \hat{i}' + \hat{i} \times) \\ &\quad + (\hat{i} \hat{i}' - \hat{i} \times) a_3(\hat{k} \times)^3 (\hat{i} \hat{i}' + \hat{i} \times) + (\hat{i} \hat{i}' - \hat{i} \times) a_4(\hat{k} \times)^4 (\hat{i} \hat{i}' + \hat{i} \times) + \dots \\ &= a_0 (\hat{i} \hat{i}' - \hat{i} \times) (\hat{i} \hat{i}' + \hat{i} \times) + a_1 (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) + a_2 (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times)^2 (\hat{i} \hat{i}' + \hat{i} \times) \\ &\quad + a_3 (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times)^3 (\hat{i} \hat{i}' + \hat{i} \times) + a_4 (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times)^4 (\hat{i} \hat{i}' + \hat{i} \times) + \dots \\ &= a_0 + a_1(\hat{j} \times) + a_2(\hat{j} \times)^2 + a_3(\hat{j} \times)^3 + a_4(\hat{j} \times)^4 + \dots \\ &= \mathbf{p}(\hat{j} \times) \end{aligned}$$

that is, the change of basis has (successfully) converted the principal  $k$ -hat axis polynomial of  $k$ -cross  $\mathbf{p}(\hat{k} \times)$  to the principal  $j$ -hat axis polynomial of  $j$ -cross  $\mathbf{p}(\hat{j} \times)$

$$\begin{aligned} \mathbf{p}(\hat{k} \times) &= a_0 + a_1(\hat{k} \times) + a_2(\hat{k} \times)^2 + a_3(\hat{k} \times)^3 + a_4(\hat{k} \times)^4 + \dots \\ (\hat{i} \hat{i}' - \hat{i} \times) \mathbf{p}(\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) &= \mathbf{p}(\hat{j} \times) \\ &\quad \downarrow \mathbf{expi} \text{ change of basis} \\ \mathbf{p}(\hat{j} \times) &= a_0 + a_1(\hat{j} \times) + a_2(\hat{j} \times)^2 + a_3(\hat{j} \times)^3 + a_4(\hat{j} \times)^4 + \dots \end{aligned}$$

and hence the change of basis from one principal axis to another works for any function that can be expanded in terms of its principal axis basis function appended with cross (here  $k \times$  and  $j \times$ ). Since the most important of such functions relate to de-Moivre's theorem, and hence a change of principal axis between the three forms of de-Moivre's theorem for each axis, I will do the maths for the  $k$ -hat exponential  $\mathbf{expk}$  and the  $k$ -hat basis sine  $\mathbf{sink}$  and cosine  $\mathbf{cosk}$  for Euler formula, which changes the principal axis of these functions to  $j$ -hat and hence converts them to  $\mathbf{expj}$ ,  $\mathbf{sinj}$ , and  $\mathbf{cosj}$ .

From here on, all of the proofs are adaptations of the proof of the change of principal axis of the arbitrary polynomial in  $k$ -cross,  $\mathbf{p}(\hat{k} \times)$ , to that in  $j$ -cross,  $\mathbf{p}(\hat{j} \times)$ . I will use the basis exponential notation rather than  $\mathbf{cisi}$  or  $\mathbf{expi}$  notation for the change of basis  $\pi/2$   $i$ -hat exponentials as these are more succinct.

4a.  $k$ -hat exponential as polynomial expansion of  $k$ -hat basis function $\exp k \theta \rightarrow \exp j \theta$  $(e^{\theta \hat{k} \times} \rightarrow e^{\theta \hat{j} \times})$ 

$$\begin{aligned}
e^{-(\pi/2)\hat{i} \times} e^{\theta \hat{k} \times} e^{(\pi/2)\hat{i} \times} &= \exp\left(-(\pi/2)\hat{i} \times\right) \exp\left(\theta \hat{k} \times (\cdot)\right) \exp\left((\pi/2)\hat{i} \times\right) \\
&= e^{-(\pi/2)\hat{i} \times} \left(1 + \theta(\hat{k} \times) + \frac{1}{2!}\theta^2(\hat{k} \times)^2 + \frac{1}{3!}\theta^3(\hat{k} \times)^3 + \frac{1}{4!}\theta^4(\hat{k} \times)^4 + \dots\right) e^{(\pi/2)\hat{i} \times} \\
&= e^{-(\pi/2)\hat{i} \times} e^{(\pi/2)\hat{i} \times} + e^{-(\pi/2)\hat{i} \times} \theta(\hat{k} \times) e^{(\pi/2)\hat{i} \times} + e^{-(\pi/2)\hat{i} \times} \frac{1}{2!}\theta^2(\hat{k} \times)^2 e^{(\pi/2)\hat{i} \times} \\
&\quad + e^{-(\pi/2)\hat{i} \times} \frac{1}{3!}\theta^3(\hat{k} \times)^3 e^{(\pi/2)\hat{i} \times} + e^{-(\pi/2)\hat{i} \times} \frac{1}{4!}\theta^4(\hat{k} \times)^4 e^{(\pi/2)\hat{i} \times} + \dots \\
&= 1 + \theta e^{-(\pi/2)\hat{i} \times} (\hat{k} \times) e^{(\pi/2)\hat{i} \times} + \frac{1}{2!}\theta^2 e^{-(\pi/2)\hat{i} \times} (\hat{k} \times)^2 e^{(\pi/2)\hat{i} \times} + \frac{1}{3!}\theta^3 e^{-(\pi/2)\hat{i} \times} (\hat{k} \times)^3 e^{(\pi/2)\hat{i} \times} + \frac{1}{4!}\theta^4 e^{-(\pi/2)\hat{i} \times} (\hat{k} \times)^4 e^{(\pi/2)\hat{i} \times} + \dots \\
&= 1 + \theta(\hat{j} \times) + \frac{1}{2!}\theta^2(\hat{j} \times)^2 + \frac{1}{3!}\theta^3(\hat{j} \times)^3 + \frac{1}{4!}\theta^4(\hat{j} \times)^4 + \dots \\
&= \exp(\theta \hat{j} \times (\cdot)) \\
&= e^{\theta \hat{j} \times}
\end{aligned}$$

that is,  $\exp\left(-(\pi/2)\hat{i} \times\right) \exp\left(\theta \hat{k} \times (\cdot)\right) \exp\left((\pi/2)\hat{i} \times\right) = \exp\left(\theta \hat{j} \times (\cdot)\right)$ , or more succinctly  $e^{-(\pi/2)\hat{i} \times} \left(e^{\theta \hat{k} \times}\right) e^{(\pi/2)\hat{i} \times} = e^{\theta \hat{j} \times}$ . This has proved the change of principal  $k$ -hat axis for the  $k$ -hat exponential **expk** to the  $j$ -hat axis converting the basis exponential to the  $j$ -hat exponential **expj**. Note here that it is wrong to say that  $e^{-(\pi/2)\hat{i} \times} \left(e^{\theta \hat{k} \times}\right) e^{(\pi/2)\hat{i} \times} = e^{-(\pi/2)\hat{i} \times + \theta \hat{k} \times + (\pi/2)\hat{i} \times} = e^{\theta \hat{k} \times}$  so this is suggestive that it may be wise, depending on circumstances, to avoid putting indices of multiplied exponentials in a common index since these do not commute with plus (which is counterintuitive to scalar mathematics, and even counter intuitive in linear algebra where matrices commute with plus +)... a possible exception being the Miller index forms for crystallography where it seems good-practice to me to put the three basis complex numbers counting rotations in the same index for a three-dimensional spatial Fourier transform [although still yet to work out the details].

The above example is whereby **expk** is rotated about the  $i$ -hat axis 90 degrees converting it to a **expj**. If the rotation is by 180 degrees around  $i$ -hat, the **expk** is converted to the **expk** conjugate (**expk\*** = **expk'** = **expk<sup>T</sup>**). If it is rotated in the opposite sense by 90 degrees (by the  $\pi/2$  **expi** conjugate) it is converted to the **expj** conjugate (**expj\*** = **expj'** = **expj<sup>T</sup>**). Similar arguments and mathematics apply to the **expi** and **expj** basis-hat exponentials, and their component sines (**sini** and **sinj**) and cosines (**cosi** and **cosj**), and their conjugates.

As complementary mathematics, in terms of matrix bases, the  $k$ -hat exponential  $\exp(\theta \hat{k} \times)$  and component harmonics are

$$\exp(\theta \hat{k} \times) = \mathbf{cisk}\theta = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\mathbf{sink}\theta) \hat{k} \times = \begin{pmatrix} 0 & -\sin \theta & 0 \\ \sin \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{cosk}\theta = \begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The  $i$ -hat exponential  $\mathbf{cisi}(\pi/2) = \exp\left(\frac{1}{2}\pi \hat{i} \times\right)$  and its conjugate  $\mathbf{cisi}(-\pi/2) = \exp\left(-\frac{1}{2}\pi \hat{i} \times\right)$  are then the matrices of the change-of-rotation-axis from  $k$ -hat to  $j$ -hat of the  $k$ -hat exponential, from  $\exp(\theta \hat{k} \times)$  to  $\exp(\theta \hat{j} \times)$ . These are

$$\mathbf{cisi}(\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{cisi}(-\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Applying the change of basis to  $\mathbf{cisk}\theta = \exp(\theta \hat{k} \times)$

$$\mathbf{cisi}(-\pi/2) \exp(\theta \hat{k} \times) \mathbf{cisi}(\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} = \mathbf{cisj}\theta = \exp(\theta \hat{j} \times)$$

which is a  $j$ -hat exponential (**expj**). Hence, the change of matrix basis (the alternative and analogous method to cross-products) has converted the  $k$ -hat exponential (**expk**) to a  $j$ -hat exponential (**expj**), and consequently, the  $k$ -hat sine (**sink**) and cosine (**cosk**) to a  $j$ -hat sine (**sinj**) and cosine (**cosj**). It all works as it should, as per complex numbers [basis curls] and as per Euler's formula which converts the quartic roots of unity to the infinite roots of unity of the sines and cosines, in the context here of the rectangular coordinate system.

**4b. Euler's formula and powers of  $k$ -hat basis exponential (de-Moivre's theorem)  $(\exp k \theta)^n = \text{cisk } n\theta \rightarrow (\exp j \theta)^n = \text{cisj } n\theta$**

This is an algebraic rearrangement of proving the change of principal axis for the basis exponential, the rearrangement showing Euler's formula ( $\exp k \theta = \text{cisk } \theta$ ) and that it also undergoes a change of principal axis to  $\exp j \theta = \text{cisj } \theta$ , that is, from  $k$ -hat to  $j$ -hat  $\exp k \theta = \text{cisk } \theta \rightarrow \exp j \theta = \text{cisj } \theta$ :

$$\begin{aligned} e^{-(\pi/2)\hat{i} \times} (\text{cisk } \theta + (\text{sink } \theta) \hat{k} \times) e^{(\pi/2)\hat{i} \times} &= e^{-(\pi/2)\hat{i} \times} e^{\theta \hat{k} \times} e^{(\pi/2)\hat{i} \times} \\ &= \exp\left(-(\pi/2)\hat{i} \times\right) \exp\left(\theta \hat{k} \times (\cdot)\right) \exp\left((\pi/2)\hat{i} \times\right) \\ &= 1 + \theta(\hat{j} \times) + \frac{1}{2!}\theta^2(\hat{j} \times)^2 + \frac{1}{3!}\theta^3(\hat{j} \times)^3 + \frac{1}{4!}\theta^4(\hat{j} \times)^4 + \frac{1}{5!}\theta^5(\hat{j} \times)^5 + \dots \\ &= \left(1 + \frac{1}{2!}\theta^2(\hat{j} \times)^2 + \frac{1}{4!}\theta^4(\hat{j} \times)^4 + \dots\right) + \left(\theta(\hat{j} \times) + \frac{1}{3!}\theta^3(\hat{j} \times)^3 + \frac{1}{5!}\theta^5(\hat{j} \times)^5 + \dots\right) \\ &= \left(1 + \frac{1}{2!}\theta^2(\hat{j} \times)^2 + \frac{1}{4!}\theta^4(\hat{j} \times)^4 + \dots\right) + \left(\theta + \frac{1}{3!}\theta^3(\hat{j} \times)^2 + \frac{1}{5!}\theta^5(\hat{j} \times)^4 + \dots\right)(\hat{j} \times) \\ &= \text{cosj } \theta + (\text{sinj } \theta) \hat{j} \times \end{aligned}$$

Hence,  $e^{-(\pi/2)\hat{i} \times} (\text{cisk } \theta + (\text{sink } \theta) \hat{k} \times) e^{(\pi/2)\hat{i} \times} = \text{cosj } \theta + (\text{sinj } \theta) \hat{j} \times$ . The **change of principal axis by change of basis** can interconvert any of the three principal forms of **Euler's formula** to any of the remaining two, that is, inter-convert between any of the three **principal basis exponentials** (for the rectangular system) and component **basis harmonics**

$$\begin{aligned} \exp\left(\theta \hat{k} \times (\cdot)\right) &= e^{\theta \hat{k} \times} = \text{cisk } \theta + (\text{sink } \theta) \hat{k} \times \\ \exp\left(\theta \hat{j} \times (\cdot)\right) &= e^{\theta \hat{j} \times} = \text{cosj } \theta + (\text{sinj } \theta) \hat{j} \times \\ \exp\left(\theta \hat{i} \times (\cdot)\right) &= e^{\theta \hat{i} \times} = \text{cosi } \theta + (\text{sini } \theta) \hat{i} \times \end{aligned}$$

The change of basis also further shows again that **expk**, **expj**, and **expi** are similar because  $k \times$ ,  $j \times$ , and  $i \times$  ( $k$ -cross,  $j$ -cross, and  $i$ -cross) are similar

$$(\text{cisk } \theta + \text{sink } \theta \hat{k} \times) \sim (\text{cosj } \theta + \text{sinj } \theta \hat{j} \times) \sim (\text{cosi } \theta + \text{sini } \theta \hat{i} \times) \quad \text{since } \hat{k} \times \sim \hat{j} \times \sim \hat{i} \times$$

that is,

$$\exp\left(\theta \hat{k} \times\right) \sim \exp\left(\theta \hat{j} \times\right) \sim \exp\left(\theta \hat{i} \times\right) \quad \text{since } \hat{k} \times \sim \hat{j} \times \sim \hat{i} \times$$

each able to be raised to powers on alternative axes by the appropriate change of basis. Ignoring the fact that there is no such thing as complex numbers or quaternions [but rather cross-products], the above mathematics means that each of the quaternion complex numbers [basis-crosses] are formally similar (in the linear algebra sense) and consequently each complex number ( $k$ -cross,  $j$ -cross, and  $i$ -cross) of the set of quaternion complex numbers are also similar.

So it is possible to make a change of basis to convert **Euler's formula** for one principal axis to that of another principal axis, and to subsequently convert **de-Moivre's theorem** ( $(\exp k \theta)^n = \text{cisk } n\theta$ ) for one principal axis (such as  $k$ -hat) to that of another principal axis, such as  $(\exp k \theta)^n = \text{cisk } n\theta$  change of base to  $(\exp j \theta)^n = \text{cisj } n\theta$ . So further for **de-Moivre's theorem**, raising to powers also holds so that if  $\text{cisi}(-\pi/2) \text{cisk } \theta \text{cisi}(\pi/2) = \text{cisj } \theta$ , i.e.,  $\text{cisi}(-\pi/2) \exp\left(\theta \hat{k} \times\right) \text{cisi}(\pi/2) = \exp\left(\theta \hat{j} \times\right)$ , then

$$(\text{cisi}(-\pi/2) \text{cisk } \theta \text{cisi}(\pi/2))^n = \text{cisi}(-\pi/2) (\text{cisk } \theta)^n \text{cisi}(\pi/2) = \text{cisi}(-\pi/2) \text{cisk } (n\theta) \text{cisi}(\pi/2) = \text{cisj } (n\theta)$$

and hence similarly for interconverting between any of

$$\begin{aligned} \exp\left(n\theta \hat{k} \times (\cdot)\right) &= e^{n\theta \hat{k} \times} = (\text{cisk } \theta + (\text{sink } \theta) \hat{k} \times)^n = \text{cisk } n\theta + (\text{sink } n\theta) \hat{k} \times \\ \exp\left(n\theta \hat{j} \times (\cdot)\right) &= e^{n\theta \hat{j} \times} = (\text{cosj } \theta + (\text{sinj } \theta) \hat{j} \times)^n = \text{cosj } n\theta + (\text{sinj } n\theta) \hat{j} \times \\ \exp\left(n\theta \hat{i} \times (\cdot)\right) &= e^{n\theta \hat{i} \times} = (\text{cosi } \theta + (\text{sini } \theta) \hat{i} \times)^n = \text{cosi } n\theta + (\text{sini } n\theta) \hat{i} \times \end{aligned}$$

$$\begin{aligned} \text{i.e.,} \quad (\exp k \theta)^n &= \text{cisk } n\theta \rightarrow (\exp j \theta)^n = \text{cisj } n\theta \\ (\exp k \theta)^n &= \text{cisk } n\theta \rightarrow (\exp i \theta)^n = \text{cisi } n\theta \end{aligned}$$

$$(\exp j\theta)^n = \text{cis} j n\theta \rightarrow (\exp k\theta)^n = \text{cis} k n\theta$$

$$(\exp j\theta)^n = \text{cis} j n\theta \rightarrow (\exp i\theta)^n = \text{cis} i n\theta$$

$$(\exp i\theta)^n = \text{cis} i n\theta \rightarrow (\exp j\theta)^n = \text{cis} j n\theta$$

$$(\exp i\theta)^n = \text{cis} i n\theta \rightarrow (\exp k\theta)^n = \text{cis} k n\theta$$

This works as per the theory of diagonalizing matrices with eigenvalues and eigenvectors, in order to raise to powers, but emphasizing interestingly, the basis-hat exponentials are **not diagonal**, nor in Jordan form, which is otherwise considered the most general form satisfying  $A = PBP^{-1}$  when raised to powers for  $A^n = PB^nP^{-1}$  which defines similarity. In terms of what was previously thought of the complex plane, the quartic roots of unity rotate by powers on each of the orthogonal Argand diagrams of a volume similarly by means of a left and right change of basis scale matrix which interconverts the two component vector bases between  $k$ -cross,  $j$ -cross, and or  $i$ -cross as the principal rotation axes. Note again importantly that  $k$ -cross,  $j$ -cross, and or  $i$ -cross and their exponentials are not diagonal [nor in a Jordan format] despite the maths working exactly as it does for diagonalized matrices [and Jordan forms as generalized diagonalized matrices].

In terms of rotational dynamics in flight, the **exp*i*** change of basis has converted a **yaw operator ( $k$ -cross)** into a **pitch operator ( $j$ -cross)** by way of a **quarter turn roll ( $i$ -cross)**. If one puts their right hand in front of them for a right-handed coordinate system with the thumb pointing up ( $k$ -hat), index finger pointing forward ( $i$ -hat), and middle finger pointing to the left ( $j$ -hat) parallel to the ground, then the  $k$ -hat exponential pivots (upon application of  $k\times$  in **exp*k***) around the thumb for yaw (scanning azimuth), the  $i$ -hat exponential pivots (upon application of  $i\times$  in **exp*i***) around the index finger for roll, and the  $j$ -hat exponential pivots (upon application of  $j\times$  in **exp*j***) around the middle finger for pitch. Applying the change of basis with **exp*i*( $\pi/2$ )** and **exp*i*( $-\pi/2$ )**, turning the hand around the forward pointing index finger until the middle finger points down and the thumb points left ( $\pi/2$  radians) is the roll of the quarter turn around  $i$ -hat. The thumb is now the  $j$ -hat axis. Pivots around the thumb (which was yaw dynamics  $k\times$  scanning the horizon) is then pitch dynamics (scanning elevation  $j\times$ ). This is physical intuition of changing the basis of the  $k$ -hat exponential **exp*k*** into that of the  $j$ -hat exponential **exp*j*** with a  $\pi/2$   $i$ -hat exponential **exp*i*** (the quarter turn roll operator to change the principal axis from  $k$ -hat to  $j$ -hat converting  $k\times$  to  $j\times$ ) which changes all of the generating  $k$ -cross ( $k\times$ ) operators [rotating around the  $k$ -hat principal axis] to  $j$ -cross ( $j\times$ ) operators [rotating around the  $j$ -hat principal axis] in the rotational basis functions.

In addition to, say, flight dynamics and extending Fourier synthesis, this will be very useful for modelling manufacturing robotic arms which have several chained rotational degrees of freedom each with rotated and translated principle axes. This ultimately corresponds to full freedom in rotational motion. So as a commonality with fluid mechanics and flight dynamics, it is where the dynamics of a robotic arm looks really "out of this world".

Since as was just shown, **exp*j*** and **exp*k*** are related by **cis*i*( $-\pi/2$ ) exp*k* $\theta$  cis*i*( $\pi/2$ ) = exp*j* $\theta$** , meaning again for purposes of argument of similarity invariants that **exp*k*** is similar to **exp*j***, in the context of **similar matrices** as per the definition of similarity in linear algebra, that is, two matrices  $A$  and  $B$  are similar ( $\sim$ ) when  $A = PBP^{-1}$ , that is,  $A \sim B$  whenever  $A = PBP^{-1}$ . By transitivity, with **exp*i*** then, **exp*k*** and **exp*i*** are also similar as are **exp*j*** and **exp*i***. This also means that **sink**, **sin*j***, and **sini** are similar, as are **cos*k***, **cos*j***, and **cosi** (and also for **tank**, **tan*j***, and **tani**, and **seck**, **sec*j***, and **seci** and so on). From the similarity definition in linear algebra, similarity infers the **similarity invariants** such as same **determinant** (det), same **trace** (tr), same **characteristic polynomial**  $\phi(\lambda)$  and same **eigenvalues**  $\lambda$  [the determinant is a bit trivial mind you seeing that the basis functions are **unitary**... still, its worth mentioning]. The interesting thing here is that whilst when in reals the characteristic polynomials are the same, the complex number ( $k$ -cross,  $j$ -cross, or  $i$ -cross) that factorizes them are different, so this may lead to a revision and update to what it means to say that two matrices are similar... these similarity invariants still hold so long as  $k$ -cross,  $j$ -cross, and  $i$ -cross are also considered similar [which is a principal result], despite being different and despite making the characteristic polynomials distinctly different [since the eigenvalues are roots for different axes]. For this and by similar logic,  $k$ -cross and  $j$ -cross are similar because the same  $\pi/2$   $i$ -hat exponential converts  $k$ -cross into  $j$ -cross by rotating for a change of principal axis with a change of basis so that  $k\times \sim j\times$  [that is,  $k$ -cross is similar ( $\sim$ ) to  $j$ -cross].

So this is relevant when a characteristic polynomial has a complex conjugate pair of roots which may look identical to another (when expanded into reals) when in fact, it is not identical but rather similar, for example, if one characteristic polynomial is a  $k\times$  ( $k$ -cross) conjugate pair ( $\pm k\times$ ) whilst the other is a  $j\times$  ( $j$ -cross) conjugate pair ( $\pm j\times$ ) for the same scalar coefficients. It may be necessary to define separate characteristic polynomials for each  $R^3$  component, deduced from a characteristic polynomial in a matrix basis. Physically, this is where **each direction component** has a **principal axis normal force** (constant and the null cross) and **two shear forces** (the crosses associated with the other two principal axes) that induce torque.

## 5a. $k$ -hat basis cosine

$$\text{cos} k\theta \rightarrow \text{cos} j\theta$$

There are several ways to do this. I do two of them, those which are most relevant [and then another in the matrix basis]. The first changes the principal axis of the component basis exponential and its conjugate in the Euler form of **cos*k*** while the second transforms the polynomial **p** in  $k$ -cross, **p( $k\times$ )**, for **cos*k***

$$\begin{aligned}
e^{-(\pi/2)\hat{i}\times}(\mathbf{cosk}\theta)e^{(\pi/2)\hat{i}\times} &= e^{-(\pi/2)\hat{i}\times}\left(\frac{1}{2}(e^{\theta\hat{k}\times} + e^{-\theta\hat{k}\times})\right)e^{(\pi/2)\hat{i}\times} \\
&= \frac{1}{2}e^{-(\pi/2)\hat{i}\times}(e^{\theta\hat{k}\times})e^{(\pi/2)\hat{i}\times} + \frac{1}{2}e^{-(\pi/2)\hat{i}\times}(e^{-\theta\hat{k}\times})e^{(\pi/2)\hat{i}\times} \\
&= \frac{1}{2}(e^{\theta\hat{j}\times} + e^{-\theta\hat{j}\times}) \\
&= \mathbf{cosj}\theta
\end{aligned}$$

$$\begin{aligned}
e^{-(\pi/2)\hat{i}\times}(\mathbf{cosk}\theta)e^{(\pi/2)\hat{i}\times} &= e^{-(\pi/2)\hat{i}\times}\left(1 + \frac{1}{2!}\theta^2(\hat{k}\times)^2 + \frac{1}{4!}\theta^4(\hat{k}\times)^4 + \dots\right)e^{(\pi/2)\hat{i}\times} \\
&= e^{-(\pi/2)\hat{i}\times}e^{(\pi/2)\hat{i}\times} + e^{-(\pi/2)\hat{i}\times}\left(\frac{1}{2!}\theta^2(\hat{k}\times)^2\right)e^{(\pi/2)\hat{i}\times} + e^{-(\pi/2)\hat{i}\times}\left(\frac{1}{4!}\theta^4(\hat{k}\times)^4\right)e^{(\pi/2)\hat{i}\times} + \dots \\
&= 1 + \frac{1}{2!}\theta^2e^{-(\pi/2)\hat{i}\times}(\hat{k}\times)^2e^{(\pi/2)\hat{i}\times} + \frac{1}{4!}\theta^4e^{-(\pi/2)\hat{i}\times}(\hat{k}\times)^4e^{(\pi/2)\hat{i}\times} + \dots \\
&= 1 + \frac{1}{2!}\theta^2(\hat{j}\times)^2 + \frac{1}{4!}\theta^4(\hat{j}\times)^4 + \dots \\
&= \mathbf{cosj}\theta
\end{aligned}$$

Hence,  $e^{-(\pi/2)\hat{i}\times}(\mathbf{cosk}\theta)e^{(\pi/2)\hat{i}\times} = \mathbf{cosj}\theta$ , that is, the change of basis has converted the principal  $k$ -hat axis of the  $k$ -hat basis cosine (**cosk**) to that of the principal  $j$ -hat axis so converting **cosk** to **cosj**.

So the similarity arguments and change of principal axis applies to the basis harmonics themselves, such as the  $k$ -hat basis cosine (**cosk**) and the  $k$ -hat basis sine (**sink**). Converting these to the  $j$ -hat basis harmonics with the change of principal axis by means of change of basis with full rank  $i$ -cross and its conjugate in terms of matrix bases is, quite simply,

$$\mathbf{cisi}(-\pi/2)\mathbf{cosk}\theta\mathbf{cisi}(\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\theta \end{pmatrix} = \mathbf{cosj}\theta$$

which has converted **cosk** to **cosj** similarly as the cross-products. These methods are equivalent.

## 5b. $k$ -hat basis sine

$$\mathbf{sink}\theta \rightarrow \mathbf{sinj}\theta$$

To avoid having to worry about the fact that  $k$ -cross is singular, I do the first bit of mathematics with Euler's formula for  $(\mathbf{sink}\theta)\hat{k}\times$  and the second bit with power series expansion for **sink**  $\theta$ :

$$\begin{aligned}
e^{-(\pi/2)\hat{i}\times}(\mathbf{sink}\theta)\hat{k}\times e^{(\pi/2)\hat{i}\times} &= e^{-(\pi/2)\hat{i}\times}\left(\frac{1}{2}(e^{\theta\hat{k}\times} - e^{-\theta\hat{k}\times})\right)e^{(\pi/2)\hat{i}\times} \\
&= \frac{1}{2}e^{-(\pi/2)\hat{i}\times}(e^{\theta\hat{k}\times})e^{(\pi/2)\hat{i}\times} - \frac{1}{2}e^{-(\pi/2)\hat{i}\times}(e^{-\theta\hat{k}\times})e^{(\pi/2)\hat{i}\times} \\
&= \frac{1}{2}(e^{\theta\hat{j}\times} - e^{-\theta\hat{j}\times}) \\
&= (\mathbf{sinj}\theta)\hat{j}\times
\end{aligned}$$

$$\begin{aligned}
e^{-(\pi/2)\hat{i}\times}(\mathbf{sink}\theta)e^{(\pi/2)\hat{i}\times} &= e^{-(\pi/2)\hat{i}\times}\left(\theta + \frac{1}{3!}\theta^3(\hat{k}\times)^2 + \frac{1}{5!}\theta^5(\hat{k}\times)^4 + \dots\right)e^{(\pi/2)\hat{i}\times} \\
&= e^{-(\pi/2)\hat{i}\times}\theta e^{(\pi/2)\hat{i}\times} + e^{-(\pi/2)\hat{i}\times}\left(\frac{1}{3!}\theta^3(\hat{k}\times)^2\right)e^{(\pi/2)\hat{i}\times} + e^{-(\pi/2)\hat{i}\times}\left(\frac{1}{5!}\theta^5(\hat{k}\times)^4\right)e^{(\pi/2)\hat{i}\times} + \dots \\
&= \theta e^{-(\pi/2)\hat{i}\times}e^{(\pi/2)\hat{i}\times} + \frac{1}{3!}\theta^3e^{-(\pi/2)\hat{i}\times}(\hat{k}\times)^2e^{(\pi/2)\hat{i}\times} + \frac{1}{5!}\theta^5e^{-(\pi/2)\hat{i}\times}(\hat{k}\times)^4e^{(\pi/2)\hat{i}\times} + \dots \\
&= \theta + \frac{1}{3!}\theta^3(\hat{j}\times)^2 + \frac{1}{5!}\theta^5(\hat{j}\times)^4 + \dots \\
&= \mathbf{sinj}\theta
\end{aligned}$$

Hence,  $e^{-(\pi/2)\hat{i}\times}(\mathbf{sink}\theta)\hat{k}\times e^{(\pi/2)\hat{i}\times} = (\mathbf{sinj}\theta)\hat{j}\times$  and  $e^{-(\pi/2)\hat{i}\times}(\mathbf{sink}\theta)e^{(\pi/2)\hat{i}\times} = \mathbf{sinj}\theta$ . The change of basis has converted the principal  $k$ -hat axis of the  $k$ -hat basis sine (**sink**) to that of the principal  $j$ -hat axis and so converting **sink** to **sinj**. With matrix bases

$$\mathbf{cisi}(-\pi/2)(\mathbf{sink}\theta\hat{k}\times)\mathbf{cisi}(\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sin\theta \\ 0 & 0 & 0 \\ -\sin\theta & 0 & 0 \end{pmatrix} = \mathbf{sinj}\theta\hat{j}\times$$

which has converted **sink** to **sinj** rationalized with  $k\times$  and  $j\times$  respectively.

Considering the basis harmonics, while the cosine in the matrix that is being operated on is written as a scalar cosine [from **cosk**], it is still dependent on a principal axis, that is, on  $k$ -hat until the complex number of **expi** in the change of basis matrix enters the matrix of **cosk** and operates on the matrix components ... the operation on **cosk** by quarter turn **expi** converts it to a **cosj** and converts the matrix basis components to those which shift the cosine to the new **cosj** positions (from the **cosk** positions). I am now of the opinion that even scalar sines and cosines are still always associated with a principal axis(?)... changed my mind on this again... back to what I had originally thought, **expk** is different though which is always associated with  $k$ -hat... but sine and cosine are always associated with an exponential... chicken or the egg? For example, I have started to consider always expanding sines and cosines in terms of their basis crosses, such as **cosk** $\theta = 1 + \frac{1}{2!}\theta^2(\hat{k} \times)^2 + \frac{1}{4!}\theta^4(\hat{k} \times)^4 + \dots$  and **cosj** $\theta = 1 + \frac{1}{2!}\theta^2(\hat{j} \times)^2 + \frac{1}{4!}\theta^4(\hat{j} \times)^4 + \dots$  for cosine and **sink** $\theta = \theta + \frac{1}{3!}\theta^3(\hat{k} \times)^2 + \frac{1}{5!}\theta^5(\hat{k} \times)^4 + \dots$  and **sinj** $\theta = \theta + \frac{1}{3!}\theta^3(\hat{j} \times)^2 + \frac{1}{5!}\theta^5(\hat{j} \times)^4 + \dots$  for sine. As in the original *k-cross and k-hat exponential* document that I wrote, the most important consideration is when the squared basis-crosses can be replaced by negatives for scalar sines and cosines for  $\sin \theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \dots$  and  $\cos \theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \dots$ . The answer is, of course, when it has operated on a complementary basis vector ( $i$ -hat or  $j$ -hat for  $k$ -cross) it reverts to a scalar sine or cosine. **It can definitely be considered as scalar** (on a particular axis [ $i$ -hat or  $j$ -hat for  $k$ -cross]), but still begs the question, is it (sine or cosine) really scalar or does it just appear that way?

Either way, **change of principal axis** with a **change of basis** of the **basis functions** expands known mathematics, most importantly, for spatial dynamics within  $R^3$ .

### Double-crosses (basis hyperbolics) - reflections

The change of bases of the basis-crosses and their exponentials (basis-exponentials) also applies to the basis-hat double-crosses. This is quite obvious [although still worth mentioning] because powers of the basis-crosses are themselves similar and hence as demonstrated transform similarly as the powers [by de-Moivre's theorem] with a change of principal axis. Hence, the three double-crosses ( $k \times k \times$ ,  $j \times j \times$ ,  $i \times i \times$ ) diadically expanded to matrix bases are similar matrices and as a consequence so are the three double-cross exponentials **expkk**, **expjj**, and **expii** [full rank reflections, i.e., basis hyperbolics]. As applies to the second order power, instead of changing the rotational axis associated with the single basis-cross the same transformation applied to the double cross changes the axis of reflection. For example, a photon of light reflected from a mirror to the left rather than straight on in front (which may be of relevance to interferometry... this theory also easily polarizes the light as well). For example in a matrix basis

$$\begin{aligned} \text{cisi}(-\pi/2) e^{\theta \hat{k} \times \hat{k} \times} \text{cisi}(\pi/2) &= \text{cisi}(-\pi/2) \text{ciskk} \theta \text{cisi}(\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\theta} \end{pmatrix} = \text{cisjj} \theta \\ &= e^{\theta \hat{j} \times \hat{j} \times} \end{aligned}$$

which changes the reflection around the  $k$ -hat axis to that around the  $j$ -hat axis. Considering that the change of basis interconverts the basis-harmonics, the basis-sines and basis-cosines, between the basis hats and between any combination for any orientation with respect to the standard basis, there are an infinite number of basis-sines and basis-cosines. The basis-sines and basis-cosines corresponding to each of the individual basis-hats, **sink**, **sinj**, **sini**, and **cosk**, **cosj**, and **cosi**, can therefore be termed the **principal basis sines and cosines for the standard rectangular basis**. Similarly with the basis-exponentials, **expk**, **expj**, and **expi**, and similarly for the double crosses (**sinkk**, **sinjj**, **sini**, **coskk**, **cosjj**, and **cosii**) and so forth.

Similarly for the complex numbers,  $k \times$ ,  $j \times$ , and  $i \times$  ( $k$ -cross,  $j$ -cross, and  $k$ -cross) can be termed the **principal complex numbers for the standard rectangular basis**. These, as complex numbers [noting again that there is no such thing as complex numbers], that is, as basis-cross-products, are later in another document extended to those for the cylindrical and spherical systems. These, which are derived in that document, are the **principal complex numbers**, the **principal sines and cosines**, and **principal basis-exponentials** for these systems, that is, for the **cylindrical and spherical basis systems**.

So this theory holds similarly for the extended set of complex operators [basis cross products] by change of basis to cylindrical and spherical systems, that is, for

**r-cross** ( $r \times$ ),  **$\theta$ -cross** ( $\theta \times$ ), and **z-cross** ( $z \times$ ), and their exponentials (**expr**, **expq**, and **expz**), in cylindrical systems (**radius-cross**, **azimuth-cross**, and **axis-cross products**)

and for

**r-cross** ( $r \times$ ),  **$\theta$ -cross** ( $\theta \times$ ), and  **$\phi$ -cross** ( $\phi \times$ ), and their exponentials (**expr**, **expq**, and **expf**) in spherical systems (**radius-cross**, **azimuth-cross**, and **colatitude-cross products**)

### Interconversion between the principal rectangular axes: reference table

The above proofs were for converting the basis functions of the principal axis of  $k$ -hat to the principal axis of  $j$ -hat. Since (1) any of the principal basis cross products ( $k$ -cross,  $j$ -cross, and  $i$ -cross) are similar and (2) the change of principal axes basis-hat exponentials

(**expk**, **expj**, and **expi**) are similar, these proofs hold for any of the basis functions for a change of basis from one principal axis to another. Since there are three principal axes in the rectangular system of  $R^3$ , any one of the principal axes can be changed to any of the other two. Therefore there are 2+2+2 changes of basis between principal axes, 2 for  $k$ -hat to either  $j$ -hat or  $i$ -hat, 2 for  $j$ -hat to either  $k$ -hat or  $i$ -hat, and 2 for  $i$ -hat to either  $k$ -hat or  $j$ -hat. The change of basis exponentials for the various changes of principal axes are here listed for reference for an arbitrary power series expansion **p** of the relevant basis-cross-product [which all the basis function are including the complex number basis-cross-product ( $k \times$ ) itself, as a single monomial term]:

- change of principal rotation axis:  $k$ -hat to  $j$ -hat  $(\hat{i}\hat{i}' - \hat{i} \times) \mathbf{p}(\hat{k} \times) (\hat{i}\hat{i}' + \hat{i} \times) = \mathbf{p}(\hat{j} \times)$
- change of principal rotation axis:  $k$ -hat to  $i$ -hat  $(\hat{j}\hat{j}' + \hat{j} \times) \mathbf{p}(\hat{k} \times) (\hat{j}\hat{j}' - \hat{j} \times) = \mathbf{p}(\hat{i} \times)$
- change of principal rotation axis:  $j$ -hat to  $k$ -hat  $(\hat{i}\hat{i}' + \hat{i} \times) \mathbf{p}(\hat{j} \times) (\hat{i}\hat{i}' - \hat{i} \times) = \mathbf{p}(\hat{k} \times)$
- change of principal rotation axis:  $j$ -hat to  $i$ -hat  $(\hat{k}\hat{k}' - \hat{k} \times) \mathbf{p}(\hat{j} \times) (\hat{k}\hat{k}' + \hat{k} \times) = \mathbf{p}(\hat{i} \times)$
- change of principal rotation axis:  $i$ -hat to  $j$ -hat  $(\hat{k}\hat{k}' + \hat{k} \times) \mathbf{p}(\hat{i} \times) (\hat{k}\hat{k}' - \hat{k} \times) = \mathbf{p}(\hat{j} \times)$
- change of principal rotation axis:  $i$ -hat to  $k$ -hat  $(\hat{j}\hat{j}' - \hat{j} \times) \mathbf{p}(\hat{i} \times) (\hat{j}\hat{j}' + \hat{j} \times) = \mathbf{p}(\hat{k} \times)$

Matrices are cross-products (complex numbers). Consequently,

*Everything that can be done with cross products (complex numbers) can be done with matrices*

and conversely

*Everything that can be done with matrices can be done with cross products (complex numbers)*

They're one-and-the-same as I will be discussing in an upcoming document that I am currently writing.

All of this can be done on a programmable calculator that can do vector and matrix maths (my calculator is essentially maple). In fact, all of the maths that I am proposing, some of which well established some of which new, will essentially **require such a calculator**. I don't think it will be long until all handheld calculators have full matrix functionality.

The most significant outcome for this, I predict with great hope, is that quantum mechanics will merge with electromagnetism and electrical engineering [similarly as complex analysis and linear algebra, and vector calculus] and this will all be done in matrices (as above) and will reach the perfection of the circuit laws with phasors (complex numbers): this will essentially be the circuit laws (which are currently for a plane) in full  $R^3$ . Working with matrices ( $R^3$  cross products) will essentially replace having to work in the phasor domain [that is, if matrix and calculus capable calculators remain reliable into the future] and will allow us to do everything we could do with complex numbers but with full quaternions, that is, full real  $R^3$  with full degrees of freedom for rotations on each of the orthogonal axes of  $R^3$  (rather than just one with  $i = \sqrt{-1}$ ). This is necessary since the vast majority of quantities in electromagnetism are associated with  $R^3$  directions. Why do I have the confidence to say this? One hurdle I was facing was trying to rationalize a complex denominator when it was a function of  $k$ -cross and  $j$ -cross. How do you do so if there is no well defined conjugate to rationalize with? It was seeming to be impossible to solve for me, I couldn't get it scalar (and before I knew what I was looking for, the expansions were too complicated since it was a double sum and the inner sum was a binomial expansion without commutation). Turns out it is easy. It is done easily with the matrix forms similarly as above. Makes short work of what was hencepreviously the impossible and can prove it all with cross-products (complex numbers) now too (working backwards from what I now know is the result I was looking for). The example I just mentioned I will be calling the **quaternion impedance** (extending the **complex impedance**) which will make possible **vector directions** on **capacitances** and **inductances**. The only hurdle for me now, is trying to get a full unification with the vector calculus. Once this is done, there will be another massive increase in the scope of applied mathematics [with, hopefully, a full unification with electromagnetism].

As a summary of what has been covered, here are the basis functions that have had the principal axis changed by means of change of basis

$$(\hat{i}\hat{i}' - \hat{i} \times)(\hat{k} \times)(\hat{i}\hat{i}' + \hat{i} \times)(\cdot) = (\hat{j} \times)(\cdot)$$

$$(\hat{i}\hat{i}' - \hat{i} \times)(\hat{k} \times)^n(\hat{i}\hat{i}' + \hat{i} \times) = (\hat{j} \times)^n$$

$$(\hat{i}\hat{i}' - \hat{i} \times)\mathbf{p}(\hat{k} \times)(\hat{i}\hat{i}' + \hat{i} \times) = \mathbf{p}(\hat{j} \times)$$

$$(\hat{i}\hat{i}' - \hat{i} \times)(e^{\theta \hat{k} \times})(\hat{i}\hat{i}' + \hat{i} \times) = e^{\theta \hat{j} \times}$$

$$(\hat{i}\hat{i}' - \hat{i} \times)(\cos \theta + (\sin \theta) \hat{k} \times)(\hat{i}\hat{i}' + \hat{i} \times) = \cos \theta + (\sin \theta) \hat{j} \times$$

$$(\hat{i}\hat{i}' - \hat{i} \times)(\cos \theta)(\hat{i}\hat{i}' + \hat{i} \times) = \cos \theta$$

$$e^{-(\pi/2)\hat{i} \times}(\hat{k} \times)e^{(\pi/2)\hat{i} \times}(\cdot) = (\hat{j} \times)(\cdot)$$

$$e^{-(\pi/2)\hat{i} \times}(\hat{k} \times)^n e^{(\pi/2)\hat{i} \times} = (\hat{j} \times)^n$$

$$e^{-(\pi/2)\hat{i} \times}\mathbf{p}(\hat{k} \times)e^{(\pi/2)\hat{i} \times} = \mathbf{p}(\hat{j} \times)$$

$$e^{-(\pi/2)\hat{i} \times}(e^{\theta \hat{k} \times})e^{(\pi/2)\hat{i} \times} = e^{\theta \hat{j} \times}$$

$$e^{-(\pi/2)\hat{i} \times}(\cos \theta)e^{(\pi/2)\hat{i} \times} = \cos \theta$$

$$\begin{aligned} (\hat{i} \hat{i}' - \hat{i} \times) ((\text{sink } \theta) \hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) &= (\text{sinj } \theta) \hat{j} \times \\ (\hat{i} \hat{i}' - \hat{i} \times) (\text{sink } \theta) (\hat{i} \hat{i}' + \hat{i} \times) &= \text{sinj } \theta \end{aligned}$$

$$\begin{aligned} e^{-(\pi/2)\hat{i} \times} ((\text{sink } \theta) \hat{k} \times) e^{(\pi/2)\hat{i} \times} &= (\text{sinj } \theta) \hat{j} \times \\ e^{-(\pi/2)\hat{i} \times} (\text{sink } \theta) e^{(\pi/2)\hat{i} \times} &= \text{sinj } \theta \end{aligned}$$

### Conjugate (negative) basis vectors

Also of relevance to the 6 different changes of principal axis (between  $k$ -hat,  $j$ -hat, and  $i$ -hat appended with cross [ $k \times$ ,  $j \times$ , and  $i \times$ ]) with change of basis is a change to a conjugate principal axis, that is, to the negative of a principal axis. I call these the neg- $k$ -hat axis, the neg- $j$ -hat axis, and the neg- $i$ -hat axis, 'neg' being a monosyllable contraction of negative. Sounds a bit strange but has proved very useful to me in the past for efficient calculator work, similarly as monosyllable integers (other than seven) ['eng' (monosyllable) being another for ten-to-the-power (5 syllables), factor of 5 more efficient... and then this is raised to further powers with complicated quantum mechanical and electrical calculations]. For a change of basis from a principal axis to a negative principal axis, this is done by taking conjugates of the change of basis matrices, that is, if the conjugate is at the front [left] putting it at the back [right] and if it is at the back [right] putting it at the front [left]. That makes 15 in all (for  $k$ -hat,  $j$ -hat, or  $i$ -hat to  $k$ -hat,  $j$ -hat,  $i$ -hat, neg- $k$ -hat, neg- $j$ -hat, or neg- $i$ -hat) that keeps the system rectangular including the somewhat trivial changes that transform from  $k$ -hat to neg- $k$ -hat,  $j$ -hat to neg- $j$ -hat, and  $i$ -hat to neg- $i$ -hat [the 3 that are reflection changes of basis from  $k$ -hat,  $j$ -hat, or  $i$ -hat]. Considering the positive and negative parts of a basis distinct, then, these are

- $k$ -hat (pos- $k$ -hat)
- $j$ -hat (pos- $j$ -hat)
- $i$ -hat (pos- $i$ -hat)
- neg- $k$ -hat
- neg- $j$ -hat
- neg- $i$ -hat

Any of these 6 can be changed to any of the 5 others. This makes 30 in total [6 reflections]. Starting with a  $k$ -hat basis function, this can be transformed to a basis function of any of the 5 of

$j$ -hat,  $i$ -hat, neg- $k$ -hat, neg- $j$ -hat, neg- $i$ -hat

Similarly starting with any of the others. 30 in total. The 6 outlined are the most important, although I keep coming across situations where it is prudent to distinguish between, say,  $k$ -hat (pos- $k$ -hat) and neg- $k$ -hat as different components within a basis system, in this case, of the rectangular system. Those that involve the neg-bases involve lots of inconvenient negatives. In a basis system I will be proposing soon, there will be no such things as negative numbers [in that system].

For changes of basis from a principal axis to a non-principal axis, the maths is similar but the equations aren't as streamlined. What they essentially end up being is a change of basis to the polar coordinate system which I am discussing in another document which I will be getting ready after this [already written most of it... still working out the particulars but there is already heaps of excellent new formulae - my first real big one was the log-normal theory of human lifespan in 2007, then derivatives in the PhD]. Two successive changes of basis to non-principal axes ends up being the polar spherical system.

### Direction ordering of the change of basis matrix and its conjugate

This current work has concentrated on changing the principal axis of the complex number [single component] cross product  $k$ -cross and the basis functions of this  $(\hat{i} \hat{i}' - \hat{i} \times) \mathbf{p}(\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) = \mathbf{p}(\hat{j} \times)$ .

For an arbitrary change of basis from one principal axis to another, the application of the change of basis matrix and the conjugate can be determined by the direction of the cross product triangle. For a change of basis rotation around  $i$ -hat to change a  $k$ -hat basis,  $i$ -hat crossed with  $k$ -hat is neg- $j$ -hat, so the conjugate  $i$ -hat exponential  $\hat{i} \hat{i}' - \hat{i} \times$  is at the front [left] and the non-conjugate  $\hat{i} \hat{i}' + \hat{i} \times$  at the back [right]

$$(\hat{i} \hat{i}' - \hat{i} \times) \mathbf{p}(\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) = \mathbf{p}(\hat{j} \times)$$

When the cross product is negative, this is reversed. For example, a change of basis around  $k$ -hat to change an  $i$ -hat basis,  $k$ -hat is crossed with  $i$ -hat is pos- $j$ -hat, so the non-conjugate  $k$ -hat exponential  $\hat{k} \hat{k}' + \hat{k} \times$  is at the front [left] and the conjugate  $\hat{k} \hat{k}' - \hat{k} \times$  at the back [right].

$$(\hat{k} \hat{k}' + \hat{k} \times) \mathbf{p}(\hat{i} \times) (\hat{k} \hat{k}' - \hat{k} \times) = \mathbf{p}(\hat{j} \times)$$

When a change a change of basis to a principal axis that is the same as the principal axis of the basis function [a  $k$ -hat change of basis matrix for **sink**, **cosk**, or **expk** for example] then the change of basis is null – it leaves the basis function unchanged. For example, a  $k$ -hat change of basis of a  $k$ -hat basis function is

$$(\hat{k}\hat{k}' - \hat{k} \times) \mathbf{p}(\hat{k} \times) (\hat{k}\hat{k}' + \hat{k} \times) = \mathbf{p}(\hat{k} \times)$$

which has operated on  $\mathbf{p}(\hat{k} \times)$  as a change of basis and, as a result, returned  $\mathbf{p}(\hat{k} \times)$  for a null effect.

### Quaternion-like cross product – change of principal axes

$k$ -cross ( $k \times$ ) was shown to be the complex number for the  $x$ - $y$  plane. It was then stated that  $j$ -cross ( $j \times$ ) was the complex number for the  $z$ - $x$  plane, and further that  $i$ -cross ( $i \times$ ) was the complex number for the  $y$ - $z$  plane. And still further it was shown that the cross product containing all three of the basis cross products [as a linear combination] was a quaternion. So what I also consider briefly here is the change of basis of principal axes applied to a quaternion like cross product [that is, three component cross product]. What happens is the component of the quaternion-like function remains unchanged whilst the other two principal axes are permuted, one with a sign change. This is similar to the cross product itself except it does not cross out the function of  $k$ -cross if it is a  $k$ -cross rotation change of principal axis, instead leaving it unchanged. For a quaternion-like cross product with  $i$ -hat axis rotation angle  $\alpha$ ,  $j$ -hat axis rotation angle  $\beta$ , and  $k$ -hat axis rotation angle  $\gamma$ , the function is

$$\mathbf{p}(\hat{i} \times, \hat{j} \times, \hat{k} \times) = \alpha \hat{i} \times + \beta \hat{j} \times + \gamma \hat{k} \times$$

Applying an  $i$ -hat change of principal axis

$$\begin{aligned} & (\hat{i} \hat{i}' - \hat{i} \times) \mathbf{p}(\hat{i} \times, \hat{j} \times, \hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) \\ &= (\hat{i} \hat{i}' - \hat{i} \times) (\alpha \hat{i} \times + \beta \hat{j} \times + \gamma \hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) \\ &= \alpha (\hat{i} \hat{i}' - \hat{i} \times) (\hat{i} \times) (\hat{i} \hat{i}' + \hat{i} \times) + \beta (\hat{i} \hat{i}' - \hat{i} \times) (\hat{j} \times) (\hat{i} \hat{i}' + \hat{i} \times) + \gamma (\hat{i} \hat{i}' - \hat{i} \times) (\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) \quad \text{Varignon's theorem} \\ &= \alpha \hat{i} \times - \gamma \hat{j} \times + \beta \hat{k} \times \end{aligned}$$

where the mathematics has utilized the distributive property of cross product. In civil and mechanical engineering [in which I became convinced that the basis vectors are the complex numbers], the distributive property of cross product is known as Varignon's theorem [Engineering Mechanics, Miriam and Kraige]. The change of basis applied to the quaternion [three component cross product] has permuted  $j$ -cross with  $k$ -cross, that is, since it is an  $i$ -hat change of principal axis, the  $i$ -hat axis rotation angle  $\alpha$  has remained unchanged at  $i$ -hat whilst the  $j$ -hat axis rotation has been turned into neg- $k$ -hat [the negative of the former  $k$ -hat rotation axis angle] and the  $k$ -hat axis rotation angle into  $j$ -hat. I am hoping and sort of expecting that I or someone who starts using this mathematics [cross-product(s) as the complex number(s)] will essentially solve Euler's equations of motion for a general solution [for principal axis rotations]. I am confident that this will provide the necessary means to solve all of the mathematical problems that have hencepreviously been out of reach for analytical solution [especially for purely scalar mathematics]. Doing pretty well so far on this actually. For a more general quaternion-like basis function, which is a linear combination of power series expansions of each of the three individual basis crosses

$$\begin{aligned} & (\hat{i} \hat{i}' - \hat{i} \times) (\alpha \mathbf{f}(\hat{i} \times) + \beta \mathbf{g}(\hat{j} \times) + \gamma \mathbf{h}(\hat{k} \times)) (\hat{i} \hat{i}' + \hat{i} \times) \\ &= \alpha (\hat{i} \hat{i}' - \hat{i} \times) \mathbf{f}(\hat{i} \times) (\hat{i} \hat{i}' + \hat{i} \times) + \beta (\hat{i} \hat{i}' - \hat{i} \times) \mathbf{g}(\hat{j} \times) (\hat{i} \hat{i}' + \hat{i} \times) + \gamma (\hat{i} \hat{i}' - \hat{i} \times) \mathbf{h}(\hat{k} \times) (\hat{i} \hat{i}' + \hat{i} \times) \\ &= \alpha \mathbf{f}(\hat{i} \times) - \gamma \mathbf{h}(\hat{j} \times) + \beta \mathbf{g}(\hat{k} \times) \end{aligned}$$

where the  $\mathbf{f}$  function of  $i$ -cross has remain unchanged whilst the  $\mathbf{g}$  function of  $j$ -cross is now a function of the  $k$ -hat principal rotation axis and the  $\mathbf{h}$  function of  $k$ -cross is now a function of the  $j$ -hat principal rotation axis with a sign change [neg- $j$ -hat principal rotation axis]. The functions  $\mathbf{f}$  can be any **cosi**, **sini**, **expi**, or **tani**, the function  $\mathbf{g}$  can be any **cosj**, **sinj**, **expj**, or **tanj** (which are converted to **cosk**, **sink**, **expk**, or **tank**) and the function  $\mathbf{h}$  can be any **cosk**, **sink**, **expk**, or **tank** (which are converted to **cosj**, **sinj**, **expj**, or **tanj**). Could also be the basis hyperbolics []basis-double- crosses for reflections such as **cosii**, **sinii**, **expii**, or **tanii** for  $\mathbf{f}$  for example.

[from my email] My next documents will be 'polar and spherical complex numbers [r-cross and the r-hat exponential]' and another I was going to call 'generalized-cross products'. Thing is, they are not generalized, so I am considering calling it instead 'the almighty cross-product'. In this, I argue that all of linear algebra is cross products. The maths for these is already [mostly] worked out and are already partially written. Thinking of writing a shorter one on moments of inertia which I will elaborate and argue that all matter is Gauss density.